

# Ricci Collineations of the Bianchi Types I and III, and Kantowski-Sachs Spacetimes

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## Abstract

Ricci collineations of the Bianchi types I and III, and Kantowski-Sachs space-times are classified according to their Ricci collineation vector (RCV) field of the form (i)-(iv) one component of  $\xi^a(x^b)$  is nonzero, (v)-(x) two components of  $\xi^a(x^b)$  are nonzero, and (xi)-(xiv) three components of  $\xi^a(x^b)$  are nonzero. Their relation with isometries of the space-times is established. In case (v), when  $\det(R_{ab}) = 0$ , some metrics are found under the time transformation, in which some of these metrics are known, and the other ones new. Finally, the family of contracted Ricci collineations (CRC) are presented.

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# 1 Introduction

The general theory of relativity is described by the Einstein field equations of the form

$$G_{ab} \equiv R_{ab} - \frac{1}{2}R g_{ab} = \kappa T_{ab} \quad (1)$$

where  $G_{ab}$  represents the components of Einstein tensor,  $R_{ab}$  are the components of the Ricci tensor and  $T_{ab}$  being the components of the energy-momentum (or matter) tensor. Here  $R(\equiv g^{ab}R_{ab})$  is the Ricci scalar and  $\kappa$  the Einstein gravitational constant. Since the Einstein tensor is related, via the Einstein field equations, to the material content of the space-time, it appears then natural to look into the symmetries of the Einstein and energy-momentum tensors. Taking the Lie derivative on both sides of equation (1), it turns out that the symmetries of the Einstein tensor, which is a function of Ricci and metric tensors, and Ricci scalar, are identically the same as the symmetries of the energy-momentum tensor. In this paper, we restrict our attention to the Ricci tensor symmetries.

According to the classification of Katzin *et al.* [1], Ricci (RC) and contracted Ricci (CRC) collineations, which are the most general symmetry transformations admitted by a given metric, are defined by

$$\mathcal{L}_\xi R_{ab} = 0, \quad g^{ab} \mathcal{L}_\xi R_{ab} = 0, \quad (2)$$

where " $\mathcal{L}_\xi$ " denotes Lie derivative along  $\xi$ . In a torsion free space in coordinate basis, the RC equations can also be written in component form as

$$R_{ab,c}\xi^c + R_{ac}\xi^c_{,b} + R_{cb}\xi^c_{,a} = 0, \quad (3)$$

where  $\xi^c$  are the components of the Ricci collineation vector (RCV) and " $,a$ " represents differentiation with respect to  $x^a$ . Each member of the family of CRC symmetry mappings satisfies  $g^{ab}\mathcal{L}_\xi(T_{ab} - g_{ab}T/2) = 0$ , which leads to the following generator for a space-time admitting RC [2]

$$\nabla \left[ (-g)^{1/2} \left( T^a_b - \frac{1}{2}\delta^a_b \right) \xi^b \right] = \partial_a \left[ (-g)^{1/2} R^a_b \xi^b \right]$$

where  $\nabla$  and  $\partial$  represent, respectively, the covariant and partial derivative operators. Also, Davies *et al.* [2] have pointed out the need for more detailed investigations of the symmetry properties lying between isometries that leave

invariant the metric tensor under Lie transport, i.e.  $\mathcal{L}_k g_{ab} = 0$ , where  $k$  is a Killing vector (KV), and RCVs, as well as for more detailed investigations of their interconnections. Space-times admitting isometries have been widely studied by Kramer *et al.* [3]

In recent years, much interest has been shown in the study of the various symmetries (particularly matter and Ricci collineations) that arise in understanding the general theory of relativity more deeply. Green *et al.* [4] and Nunez *et al.* [5] have considered an example of RC and the family of CRC symmetries of Robertson-Walker metric, and they have confined their study to symmetries generated by the vector fields of the following form, respectively,

$$\xi = \xi^4(r, \theta, \phi, t) \frac{\partial}{\partial t} \quad \text{and} \quad \xi = \xi^1(r, t) \frac{\partial}{\partial r} + \xi^4(r, t) \frac{\partial}{\partial t}.$$

Also, the relationship with constants of motion between RC and family of CRC has been indicated [6]-[9]. A complete classification of RCs was obtained for spherically symmetric static spacetimes [10], which were compared with KVs admitted by the corresponding spacetimes. Amir *et al.* [11] have investigated the relationship between the RCVs and the KVs for these spacetimes in detail. Carot *et al.* [12] have studied matter collineations, as a symmetry property of the energy-momentum tensor  $T_{ab}$  and also, Hall *et al.* [13] have presented a discussion of Ricci and matter collineations in space-time.

In recently, we have discussed that RCs and family of CRCs of Bianchi type- II, VIII, and IX spacetimes by considering some RCVs [14]. Now, we consider that the metric for Bianchi types I( $\delta = 0$ ) and III( $\delta = -1$ ), and Kantowski-Sachs ( $\delta = +1$ ) cosmological models is in the form [15]-[17],

$$ds^2 = dt^2 - A^2 dr^2 - B^2(d\theta^2 + f^2 d\phi^2), \quad (4)$$

which is spatially homogeneous, has shear, and has no rotation; where  $A$  and  $B$  are functions of  $t$  only, and for  $\delta = 0, -1, +1$ , respectively,

$$f(\theta) = \begin{bmatrix} \theta \\ \sinh \theta \\ \sin \theta \end{bmatrix} \quad (5)$$

(prime denotes derivative with respect to  $\theta$ ). Further, we observe that

$$f^2 \left( \frac{f'}{f} \right)' = -1 \quad \Leftrightarrow \quad (f f')' = 2f'^2 - 1. \quad (6)$$

The nonvanishing Ricci tensor components of the above metric are as follows

$$\begin{aligned} R_{11} &= A^2 \left[ \frac{\ddot{A}}{A} + 2 \frac{\dot{A}\dot{B}}{AB} \right], \\ R_{22} &= B^2 \left[ \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \left( \frac{\dot{B}}{B} \right)^2 + \frac{\delta}{B^2} \right], \\ R_{33} &= f^2 R_{22}, \\ R_{44} &= - \left[ \frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} \right], \end{aligned}$$

where the dot represents derivative with respect to time. The  $\delta = 0$  Bianchi type I model clearly contains the flat Robertson-Walker model as a particular case, but the other two are eternally anisotropic in the sense that sectional curvatures are never simultaneously identical.

In this article, we investigate the symmetry properties of the Bianchi types I and III, and Kantowski-Sachs space-times by considering RCVs associated to the following vector fields  $\xi$  : (i)-(iv) one component of  $\xi^a(x^b)$  is nonzero, (v)-(x) two components of  $\xi^a(x^b)$  are nonzero, (xi)-(xiv) three components of  $\xi^a(x^b)$  are nonzero, where  $x^a = (r, \theta, \phi, t)$  for  $a = 1, 2, 3, 4$ . Then, we substitute these collineations into each of the collineation equations. Some of these equations, in this process, are identically satisfied, whereas the others are not and get replaced by a set of partial differential equations to be solved for classifying collineations. These collineations are explicitly derived and classified completely in section 3, while section 2 presents some general results and sets up the distinction between nondegenerate and degenerate Ricci tensors. In section 4, the family of CRCs will be briefly discussed. Finally, in the last section, we conclude with a brief summary and discussion of results of the present work.

## 2 some general results

Let  $\xi$  be a RC, i.e. a solution of equation (3) for  $\xi$ . The Lie algebra of RCs may be finite or infinite dimensional. Then, it would be interesting to know under what conditions the Lie algebra of RCs on a space-time manifold is finite dimensional. When studying the Lie algebra of RCs, two cases arise

naturally according to whether  $R_{ab}$  is nondegenerate, or degenerate. While the Ricci tensor must also have a noninfinite determinant, it can be zero [11]. In recent paper's of Hall et al. [13], they have noticed some important results the following.

(i) If  $R_{ab}$  is degenerate, then  $\text{rank}(R_{ab}) < 4$  (i.e.,  $\det(R_{ab}) = 0$ ), and we cannot guarantee the finite dimensionality of the corresponding Lie algebra. If  $R_{ab}$  is nondegenerate, then  $\text{rank}(R_{ab}) = 4$ , and Lie algebra of RCs is finite-dimensional ( $\leq 10$ , and  $\neq 9$ ).

(ii) Assuming that  $\psi$  is a (smooth) scalar function on space-time manifold, then from equation (3), yields

$$\mathcal{L}_X R_{ab} = \mathcal{L}_{\psi \xi} R_{ab} = \psi_{,b} R_{ac} \xi^c + \psi_{,a} R_{cb} \xi^c. \quad (7)$$

Therefore, it follows from eq. (7) that  $X = \psi \xi$  is also a RC if and only if either  $\psi$  is constant on the manifold or  $\xi$  satisfies  $R_{ab} \xi^b = 0$ , in which case  $R_{ab}$  is necessarily degenerate (i.e., its rank is less than 4). That is, in the latter case, it follows that if this is the case for  $\xi \neq 0$ , then the vector space of RCs on space-time manifold is infinite dimensional.

### 3 Ricci Collineations

For the Bianchi type I and III, and Kantowski-Sachs spacetimes, substituting  $R_{\alpha\alpha}$  ( $\alpha = 1, 2, 4$ ) into Eq. (3) we obtain the following RC equations :

$$\dot{R}_{11} \xi^4 + 2R_{11} \xi_{,1}^1 = 0, \quad (8)$$

$$\dot{R}_{22} \xi^4 + 2R_{22} \xi_{,2}^2 = 0, \quad (9)$$

$$f \dot{R}_{22} \xi^4 + 2R_{22} (f' \xi^2 + f \xi_{,3}^3) = 0, \quad (10)$$

$$\dot{R}_{44} \xi^4 + 2R_{44} \xi_{,4}^4 = 0, \quad (11)$$

$$R_{11} \xi_{,2}^1 + R_{22} \xi_{,1}^2 = 0, \quad (12)$$

$$R_{11} \xi_{,3}^1 + f^2 R_{22} \xi_{,1}^3 = 0, \quad (13)$$

$$R_{11} \xi_{,4}^1 + R_{44} \xi_{,1}^4 = 0, \quad (14)$$

$$R_{22} (\xi_{,3}^2 + f^2 \xi_{,2}^3) = 0, \quad (15)$$

$$R_{22} \xi_{,4}^2 + R_{44} \xi_{,2}^4 = 0, \quad (16)$$

$$f^2 R_{22} \xi_{,4}^3 + R_{44} \xi_{,3}^4 = 0. \quad (17)$$

where commas denote the partial derivatives, and the indices 1, 2, 3, and 4 correspond to the variables  $x, y, z$  and  $t$ , respectively. Note that  $R_{33}$  does not appear in the above equations, since  $R_{33} = f^2 R_{22}$ .

The nature of solution of RC equations (8)-(17) changes if one (or more) of the components of the symmetry vector  $\xi^a(x^b)$  is zero. In the above equations (8)-(17) we will consider all subcases (i)-(xiv) of the general RCV  $\xi^a$  given above section.

We first consider the simpler cases when one component of  $\xi^a(x^b)$  is different from zero. We now consider case (i),  $\xi^1(x^b) \neq 0$ . For this case,  $\xi^1$  becomes constant. In case (ii),  $\xi^2(x^a) \neq 0$ , using RC equations we obtain that  $f' R_{22} \xi^2 = 0$ , that is, either  $f = \text{const.}$  or  $R_{22} = 0$ . In the latter case,  $\xi^2$  is unconstrained function of  $x^a$ , and in the former case  $\xi^2$  is a constant. Also, the remaining equations become identities. In case (iii),  $\xi^3(x^a) \neq 0$ , using RC equations we find that  $\xi^3$  is a constant, while the other RC equations become identities. For case (iv),  $\xi^4(x^a) \neq 0$ , it follows from RC equations that  $R_{11}$  and  $R_{22}$  are constants (so that  $R_{33} = \text{const.} \times f^2$ ), and

$$\xi^4(t) = \frac{c}{\sqrt{|R_{44}|}},$$

where  $c$  is an integration constant.

Now, we obtain the other all subcases with two and three RCV components.

**Case (v):**  $\xi = (\xi^1(x^a), \xi^2(x^a), 0, 0)$ .

In this case, from equation (10) we find that  $R_{22} \xi^2 = 0$ , Then there are two possibilities (a)  $R_{22} = 0$ ; (b)  $\xi^2 = 0$ . Hence, using the other RC equations, the subcase (b) is reduced to the case (i) if  $R_{11} \neq 0$ . In this subcase, if  $R_{11} = 0$ , then  $\xi^1$  is unconstrained function of  $x^a$ , i.e. RCV field is  $\xi = \xi^1(x^a) \partial / \partial r$ . In subcase (a), i.e. when  $R_{22} = 0$ ,  $\xi^2$  is unconstrained function of  $x^a$ , and  $R_{11} \xi_{,a}^1 = 0$  for  $a = 1, 2, 3, 4$ . Thus, when  $\xi_{,a}^1 = 0$ , then  $\xi^1$  is a constant, that is  $\xi = \text{const.} \times \partial / \partial r + \xi^2(x^a) \partial / \partial \theta$ . When  $R_{11} = 0$ , then  $\xi^1$  is unconstrained function of  $x^a$  (i.e.,  $\xi = \xi^1(x^a) \partial / \partial r + \xi^2(x^a) \partial / \partial \theta$ ).

Thus, using the Ricci tensor components in which the case both  $R_{11}$  and  $R_{22}$  are zero, we get the following differential equations

$$\frac{\ddot{A}}{A} + 2 \frac{\dot{A} \dot{B}}{A B} = 0, \quad (18)$$

$$\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right) + \frac{\delta}{B^2} = 0. \quad (19)$$

For the Bianchi type I ( $\delta = 0$ ), under the general time transformation  $dt = h(A, B)d(\text{newtime})$  where  $h$  is an arbitrary functions of  $A$  and  $B$ , using Eqs. (18) and (19), we find the solution in the form  $A = (at+b)^{m/(m+2)}$ ,  $B = A^{1/m}$ , where  $m$  is a constant. Thus, in this case, the metric (4) becomes

$$ds^2 = dt^2 - (at+b)^{2m/(m+2)} dr^2 - (at+b)^{2/(m+2)} (d\theta^2 + \theta^2 d\phi^2). \quad (20)$$

This metric is a one-parameter stiff perfect fluid LRS Bianchi type I solutions of the Einstein field equations [3].

Furthermore, from Eqs. (18) and (19), we find that

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 3\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right) + \frac{\delta}{B^2} = 0$$

which leads to

$$[B(AB)]' = -\delta A. \quad (21)$$

When we use the transformation of the time coordinate in Eq. (21), we obtain the following general solution for the Bianchi type I ( $\delta = 0$ ) and III ( $\delta = -1$ ), and Kantowski-Sachs ( $\delta = +1$ ) metrics,

$$(AB)^2 = -\delta \bar{t}^2 + b_1 \bar{t} + b_2, \quad (22)$$

where  $b_1$  and  $b_2$  are constants. Further, under the change of the time-coordinate by  $dt = B^2 d\tau$ , Eqs. (18) and (19) are transformed into

$$\frac{A_{\tau\tau}}{A} = 0, \quad (23)$$

$$\frac{B_{\tau\tau}}{B} - \left(\frac{B_\tau}{B}\right)^2 + \frac{A_\tau B_\tau}{AB} + \delta B^2 = 0, \quad (24)$$

where subscript represents derivative with respect to  $\tau$ . Now, from Eq. (23), we get

$$A(\tau) = c_1 \tau + c_2.$$

In the case of  $\delta = 0$  (i.e., Bianchi type I case), the solution of the equation (24), gives

$$B(\tau) = (c_1 \tau + c_2)^{c_3/c_1},$$

where  $c_1, c_2$ , and  $c_3$  are constants. When  $\delta = \pm 1$  (Bianchi III, and Kantowski-Sachs) and  $c_1 = 0$  (i.e.,  $A = \text{const.}$ ), then for the Bianchi type III and Kantowski-Sachs space-times, from eq. (24), we obtain the following solutions, respectively,

$$B(\tau) = c_4 \sec(c_4(\tau - c_5)),$$

$$B(\tau) = c_6 / \cosh(c_7\tau),$$

where  $c_4, c_5, c_6$ , and  $c_7$  are constants. Therefore, using the above equalities in  $dt = B^2 d\tau$ , we find that  $t = \tan(c_4\tau + c_5)$ , and  $t = (c_6^2/c_7) \tanh(c_7\tau)$ , respectively.

**Case (vi).**  $\xi = (\xi^1(x^a), 0, \xi^3(x^a), 0)$ .

In this case, from RC equations, we find that  $\xi^1$  and  $\xi^3$  are functions of  $\phi$  and  $r$ , respectively, and Eqs. (9), (11), and (16) become identities if  $R_{11}$  and  $R_{22}$  are not vanished. Using the remaining equation (13), after some algebraic manipulation, we find

$$\xi_{(1)} = \frac{\partial}{\partial r}, \quad \xi_{(2)} = \frac{\partial}{\partial \phi},$$

which are the KV fields.

**Case (vii):**  $\xi = (0, \xi^2(x^a), \xi^3(x^a), 0)$ .

From Eqs. (9), (12) and (16), it follows that  $\xi^2 = \xi^2(\phi)$ , and from Eqs. (13) and (17),  $\xi^3 = \xi^3(\theta, \phi)$ . When  $R_{22} \neq 0$ , after a systematic integration of the equations (10) and (14) it is shown that RCV field is

$$\xi = -c_1 \left( \sin\phi \frac{\partial}{\partial \theta} + \frac{f'}{f} \cos\phi \frac{\partial}{\partial \phi} \right) + c_2 \left( \cos\phi \frac{\partial}{\partial \theta} - \frac{f'}{f} \sin\phi \frac{\partial}{\partial \phi} \right) + c_3 \frac{\partial}{\partial \phi}.$$

If we set the constants  $c_1, c_2$ , and  $c_3$  into 0 or  $\pm 1$ , we find the following generators of a group  $G_3$ ;

$$\xi_{(1)} = \sin\phi \frac{\partial}{\partial \theta} + \frac{f'}{f} \cos\phi \frac{\partial}{\partial \phi}, \quad \xi_{(2)} = \cos\phi \frac{\partial}{\partial \theta} - \frac{f'}{f} \sin\phi \frac{\partial}{\partial \phi}, \quad \xi_{(3)} = \frac{\partial}{\partial \phi}.$$

These are just the KV fields associated with spherical symmetry of the Bianchi types I and III, and Kantowski-Sachs metrics for  $\delta = 0, -1$ , and  $+1$ , respectively [15].

**Case (viii).**  $\xi = (\xi^1(x^a), 0, 0, \xi^4(x^a))$ .



From the RC equations (9) or (10), it follows that  $\dot{R}_{22}\xi^4 = 0$ ; thus, there are two possibilities: (I)  $\dot{R}_{22} = 0$ , i.e.,  $R_{22}$  is a constant; (II)  $\xi^4 = 0$ . In the first subcase, when  $R_{11}$  and  $R_{44}$  are not zero, then  $\xi^1$  and  $\xi^4$  depend on  $r$  and  $t$  only. For this subcase, Eq. (11) yields

$$\xi^4(r, t) = \frac{A_1(r)}{\sqrt{|R_{44}|}}, \quad (25)$$

where  $A_1(r)$  is an integration function over the variable  $r$ . Incorporating (25) into (8), and integrating the resulting equation with respect to  $r$ , we obtain

$$\xi^1(r, t) = -\frac{\ddot{R}}{2R_{11}\sqrt{|R_{44}|}} \int A_1(r)dr + A_2(t), \quad (26)$$

where  $A_2(t)$  is an integration function. Thus, using (25) and (26) in eq. (14), gives

$$\left[ \frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} \right] \cdot \int A_1 dr = \frac{\sqrt{|R_{44}|}}{R_{11}} A_{1,r} + \dot{A}_2, \quad (27)$$

where subscript represents derivative with respect to  $r$ . Differentiating the above equation relative to  $r$ , provided that  $A_1(r) \neq 0$ , yields

$$\frac{A_{1,rr}}{A_1} = \frac{R_{11}}{\sqrt{|R_{44}|}} \left[ \frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} \right] \cdot = \alpha, \quad (28)$$

where  $\alpha$  is a constant of separation. In this case, there are three possibilities: (a)  $\alpha > 0$ ; (b)  $\alpha < 0$ ; (c)  $\alpha = 0$ . From equations (25)-(28), the results in the case (a) are given in the following form :

$$\begin{aligned} \xi^1 &= -\frac{\dot{R}_{11}}{2\sqrt{\alpha}R_{11}\sqrt{|R_{44}|}} [c_1 \sinh r + c_2 \cosh r] + c_3, \\ \xi^4 &= \frac{c_1 \cosh(\sqrt{\alpha}r) + c_2 \sinh(\sqrt{\alpha}r)}{\sqrt{|R_{44}|}}, \end{aligned}$$

where  $c_1, c_2$  and  $c_3$  are constants. The solutions for case (b) are similar to case (a) but " $\alpha$ " is replaced by " $-\alpha$ " and hyperbolic functions by trigonometric ones.

For subcase (c), we get

$$\frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} = \beta, \quad (29)$$

where  $\beta$  is an integration constant. Now, there are two cases: ( $\star$ )  $\beta \neq 0$ ; ( $\dagger$ )  $\beta = 0$ . In these subcases, the corresponding RCVs are as follows, respectively,

$$\xi^1 = -\beta \left( c_1 \frac{r^2}{2} + c_2 r \right) - c_1 \int \frac{\sqrt{|R_{44}|}}{R_{11}} dt + c_3, \quad \xi^4 = \frac{c_1 r + c_2}{\sqrt{|R_{44}|}}$$

and

$$\xi^1 = \frac{c_1}{R_{11}} \int \sqrt{|R_{44}|} dt + c_2, \quad \xi^4 = \frac{c_1 r + c_2}{\sqrt{|R_{44}|}}, \quad (R_{11} = \text{const.})$$

where  $c_1, c_2, c_3$  are constants. The remaining possible case is that  $R_{11}$  and  $R_{44}$  do not satisfy equation (28). In this case  $A_1 = 0$  and resulting RCV is same as the case (i). Also, in the second subcase (II), RCVs are reduced the case (i).

$$\begin{aligned} \xi_{(1)} &= \frac{\partial}{\partial \theta}, \\ \xi_{(2)} &= \frac{1}{\sqrt{|R_{44}|}} \frac{\partial}{\partial t}. \end{aligned}$$

**Case (ix).**  $\xi = (0, \xi^2(x^a), 0, \xi^4(x^a))$ .

In this case, from equation (8), we find that there are two possibilities: (I)  $\dot{R}_{11} = 0$ , i.e.  $R_{11} = \text{const.}$ ; (II)  $\xi^4 = 0$ . The latter case is reduced to the case (ii). In subcase (ix.I), from eqs. (12), (14), (15), and (17), we obtain that  $\xi^2$  and  $\xi^4$  emerge as arbitrary functions of  $\theta$  and  $t$ . Thus, eq. (11) yields

$$\xi^4(\theta, t) = \frac{B_1(\theta)}{\sqrt{|R_{44}|}}, \quad (30)$$

where  $B_1(\theta)$  is an integration function, and  $R_{44} \neq 0$ . Subtracting Eq. (9) from Eq. (10), and integrating the resulting equation with respect to  $\theta$ , gives

$$\xi^2(\theta, t) = f B_2(t), \quad (31)$$

where  $f$  take values  $\theta, \sinh\theta, \sin\theta$  for the Bianchi type *I* and *III*, and Kantowski-Sachs space-times, respectively, and  $B_2(t)$  is an integration function. Thus, using eq. (6), it follows that

$$\frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}} \frac{1}{B_2(t)} = -\frac{f'}{B_1(\theta)} = \gamma, \quad (32)$$

where  $\gamma$  is a separation constant which is different from zero but it can take the value of 1 without losing of generality. Thus, from the constraint (32), we are left with

$$B_1(\theta) = -f', \quad B_2(t) = \frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}}, \quad (33)$$

Therefore, the components  $\xi^2$  and  $\xi^4$  are

$$\xi^2 = \frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}}, \quad \xi^4 = -\frac{f'}{\sqrt{|R_{44}|}}. \quad (34)$$

Thus, using (5), it follows from eq.(16) that the other constraint equation is

$$\frac{R_{22}}{\sqrt{|R_{44}|}} \left[ \frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}} \right] = \frac{f''}{f} = -\delta, \quad (35)$$

where  $\delta$  take values of 0 (Bianchi I),  $-1$  (Bianchi III), or  $+1$  (Kantowski-Sachs). In the case of  $\delta = 0$  (Bianchi I), from the constraint (35), we get

$$\frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}} = \eta, \quad (36)$$

where  $\eta$  is an integration constant. In the case of  $\eta = 0$ , it reduce to the case (iv). If  $\eta \neq 0$ , then the RCVs are

$$\xi = \eta\theta \frac{\partial}{\partial\theta} + \frac{1}{\sqrt{|R_{44}|}} \frac{\partial}{\partial t}.$$

**Case (x).**  $\xi = (0, 0, \xi^3(x^a), \xi^4(x^a))$ .

In this case, it follows from Eqs. (8) and (9) that  $R_{11}$  and  $R_{22}$  are constants (when  $\xi^4 = 0$ , this case is reduced to the case (ii)). From Eqs. (13) and (15), and using the last result in Eq. (10), we find that  $\xi^3$  is a function of  $t$  only, while from Eqs. (14) and (16), we have that  $\xi^4$  is functions of  $\phi$  and  $t$ . Also, Eq. (11) gives

$$\xi^4(\theta, t) = \frac{D(\phi)}{\sqrt{|R_{44}|}}, \quad (37)$$

where  $D(\phi)$  is a function of integration with respect to  $t$ . Plugging this equation into Eq. (17) and differentiating, we obtain the following RCVs

$$\xi_{(1)} = \frac{\partial}{\partial\theta}, \quad \xi_{(2)} = \frac{1}{\sqrt{|R_{44}|}} \frac{\partial}{\partial t}.$$

**Case (xi).**  $\xi = (\xi^1(x^a), \xi^2(x^a), \xi^3(x^a), 0)$ .

In this case, from eqs. (8), (9), (14), and (17), it follows that  $\xi^1 = \xi^1(\theta, \phi)$ ,  $\xi^2 = \xi^2(r, \theta)$ , and  $\xi^3 = \xi^3(r, \theta, \phi)$ . Then, from eq. (12), yields

$$\begin{aligned} \xi^1 &= A_1(\phi)\theta + A_2(\phi), \\ \xi^2 &= -aA_1(\phi)r + A_3(\phi), \\ R_{11} &= aR_{22}, \end{aligned}$$

where  $a$  is a constant;  $A_1, A_2$ , and  $A_3$  are integration constants. Using above mentioned results in eq. (13), we find

$$\xi^3 = \frac{ar}{f^2} (A_{1,\phi}\theta + A_{2,\phi}) + B_1(\theta, \phi)$$

where  $B_1(\theta, \phi)$  is a constant of integration. If  $a = 0$ , i.e.  $R_{11} = 0$ , then  $\xi^1$  is unconstrained function of  $x^a$ , and the other components are the same as in the case (vii). Clearly the vector space of RCVs is infinite dimensional in the latter case. When  $a \neq 0$  and  $\delta = 0$  (Bianchi I), then substituting  $\xi^2$  and  $\xi^3$  into eq. (10), after some algebra, we obtain that

$$\begin{aligned} A_1 &= c_1 \cos\phi + c_2 \sin\phi, \\ A_2 &= c_3 \phi + c_4, \\ B_1 &= -\frac{1}{\theta} \int A_3 d\phi + A_4(\theta), \end{aligned}$$

where  $c_i$  ( $i = 1, 2, 3, 4$ ), are constants, and  $A_4$  is a new integration function. Thus, we find from eq. (15) that  $c_3$  vanishes, and also  $A_3$  and  $A_4$  take the following form

$$\begin{aligned} A_3 &= c_5 \cos\phi + c_6 \sin\phi, \\ A_4 &= c_7, \end{aligned}$$

where  $c_5, c_6$ , and  $c_7$  are constants. Therefore, relabelling  $c_4 \leftrightarrow c_3, c_5 \leftrightarrow c_4, c_6 \leftrightarrow c_5$ , and  $c_7 \leftrightarrow c_6$ , and setting these parameters into 0 or  $\pm 1$ , we obtain  $R_{11} = aR_{22}$  and six collineations

$$\xi_{(3)} = \frac{\partial}{\partial\phi}, \quad \xi_{(4)} = \theta \sin\phi \frac{\partial}{\partial r} - ar\xi_{(1)}, \quad \xi_{(5)} = \theta \cos\phi \frac{\partial}{\partial r} - ar\xi_{(2)}, \quad \xi_{(6)} = \frac{\partial}{\partial r},$$

where  $a$  is a nonzero constant, and  $\xi_{(1)}$  and  $\xi_{(2)}$  are given in the case (vii). The Lie algebra of RCVs is spanned by these vectors. The vectors  $\xi_{(1)}, \xi_{(2)}$ , and  $\xi_{(3)}$  correspond to the KV fields associated with spherical symmetry, while the other ones are the proper RCVs of the Bianchi type I space-time. The nonvanishing commutators are given by the following

$$\begin{aligned} [\xi_{(1)}, \xi_{(3)}] &= -\xi_{(2)}, & [\xi_{(1)}, \xi_{(4)}] &= [\xi_{(2)}, \xi_{(5)}] = \xi_{(6)}, \\ [\xi_{(2)}, \xi_{(3)}] &= \xi_{(1)}, & [\xi_{(2)}, \xi_{(6)}] &= [\xi_{(3)}, \xi_{(4)}] = \xi_{(5)}, & [\xi_{(3)}, \xi_{(5)}] &= -\xi_{(4)}, \\ [\xi_{(4)}, \xi_{(5)}] &= a\xi_{(3)}, & [\xi_{(4)}, \xi_{(6)}] &= a\xi_{(1)}, & [\xi_{(5)}, \xi_{(6)}] &= a\xi_{(2)}, \end{aligned}$$

If  $a \neq 0$  and  $\delta \neq 0$  (for Bianchi III and Kantowski-Sachs), then  $A_1$  vanishes, and  $A_2, A_3$ , and  $A_4$  are same as the above one. Thus, three of the four collineations remain the same as in case (vii), whereas the other is  $\xi_{(4)} = \partial/\partial r$ . These are nonproper RCVs and the generators of the group  $G_4$ .

**Case (xii).**  $\xi = (\xi^1(x^a), 0, \xi^3(x^a), \xi^4(x^a))$ .

In this case, after a serial algebra, it follows from the resulting RC equations that the constraints do appear in a similar way as in case (viii), such as the eqs. (28) and (29), and hence the components  $\xi^1$  and  $\xi^4$  keep the forms as in case (viii), and whereas  $\xi^3$  becomes a constant.

**Case (xiii).**  $\xi = (\xi^1(x^a), \xi^2(x^a), 0, \xi^4(x^a))$ .

In this case, using the RC eqs. (13), (15), and (17), then the components  $\xi^1, \xi^2$ , and  $\xi^4$  of RCV emerge as an arbitrary functions of  $r, \theta$ , and  $t$ . Then from eq. (11) yields

$$\xi^4 = \frac{A_1(r, \theta)}{\sqrt{|R_{44}|}},$$

where  $A_1(r, \theta)$  is an integration function. Putting this component into eq. (10), gives

$$\xi^2 = -\frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}} \frac{A_1 f}{f'},$$

Thus, inserting  $\xi^2$  and  $\xi^4$  into eq. (9), and integrating, we obtain

$$A_1 = f' B_1(r), \quad (38)$$

regardless of whether  $\dot{R}_{22} = 0$  or not. Here,  $B_1(r)$  is a function of integration. Substituting  $\xi^4$  into eq.(8), and using (38), we find that

$$\xi^1 = -\frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} \int B_1(r) dr + A_2(\theta, t),$$

where  $A_2$  is an arbitrary function of the integration over the variable  $r$ . Then, using  $\xi^2$  and  $\xi^4$  in (16),  $\xi^1$  and  $\xi^4$  in (12), and differentiating, we obtain the following constraints, respectively,

$$\frac{R_{22}}{\sqrt{|R_{44}|}} \left[ \frac{\dot{R}_{22}}{2R_{22}\sqrt{|R_{44}|}} \right] = \frac{f''}{f} = -\delta,$$

$$\frac{R_{11}}{\sqrt{|R_{44}|}} \left[ \frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} \right] = \frac{B_{1,rr}}{B_1} = \alpha,$$

where  $\delta$  take values of  $0, \pm 1$  for Bianchi type I and III, and Kantowski-Sachs space-times and  $\alpha$  is a separation constant. Thus, there are six different cases: A)  $\alpha > 0$  and  $\delta \neq 0$ , B)  $\alpha < 0$  and  $\delta \neq 0$ , C)  $\alpha = 0$  and  $\delta \neq 0$ , D)  $\alpha > 0$  and  $\delta = 0$ , E)  $\alpha < 0$  and  $\delta = 0$ , F)  $\alpha = 0 = \delta$ .

In subcase A), one finds  $B_1(r) = c_1 \cosh r + c_2 \sinh r$ , and substituting this result into  $\xi^1$  and  $\xi^2$ , integrating and plugging them into (12), after some algebra, we find

$$\xi^1 = -\frac{f' \dot{R}_{11}}{2\sqrt{\alpha} R_{11} \sqrt{|R_{44}|}} \left[ c_1 \sinh(\sqrt{\alpha} r) + c_2 \cosh(\sqrt{\alpha} r) \right] + c_3,$$

$$\xi^2 = -\frac{f \dot{R}_{11}}{2\sqrt{\alpha} R_{11} \sqrt{|R_{44}|}} \left[ c_1 \cosh(\sqrt{\alpha} r) + c_2 \sinh(\sqrt{\alpha} r) \right],$$

$$\xi^4 = \frac{f'}{\sqrt{|R_{44}|}} \left[ c_1 \cosh(\sqrt{\alpha} r) + c_2 \sinh(\sqrt{\alpha} r) \right],$$

$$R_{22} = \frac{\delta}{\alpha} R_{11} + a, \quad a = \text{const.}$$

where  $\delta$  take values of  $\pm 1$  only. In subcase B), the solutions are similar to the subcase A) but " $\alpha$ " is replaced by " $-\alpha$ " and hyperbolic functions by trigonometric ones. In subcase C), we get

$$B_1(r) = d_1 r + d_2, \quad \frac{\dot{R}_{11}}{2R_{11}\sqrt{|R_{44}|}} = \beta$$

where  $d_1, d_2$  are arbitrary constants, and  $\beta$  is a constant of integration. Thus, in this subcase, there are also two possibilities : a)  $\beta \neq 0$ , b)  $\beta = 0$ . First one is reduced to the case (i). In latter one, RCVs and the restrictions are as follows

$$\xi = \left( \frac{d_1 \dot{R}_{22}}{2R_{11}\sqrt{|R_{44}|}} \int f d\theta + d_3 \right) \frac{\partial}{\partial r} - \frac{f \dot{R}_{22}(d_1 r + d_2)}{2R_{22}\sqrt{|R_{44}|}} \frac{\partial}{\partial \theta} + \frac{f'(d_1 r + d_2)}{\sqrt{|R_{44}|}} \frac{\partial}{\partial t},$$

$$\begin{aligned}
\frac{\dot{R}_{22}}{\sqrt{|R_{44}|}} &= -\delta \int \sqrt{|R_{44}|} dt + \text{const.} \\
(R_{22} - 1) \sqrt{|R_{44}|} &= \text{const.}, \\
R_{11} &= \text{const.},
\end{aligned}$$

In subcase D) and E), it follows from RC equations that  $\xi^2$  is a constant, i.e.; this case is reduced to the case (i). Finally, in the last subcase F), i.e. when  $f(\theta) = \theta$ -Bianchi I,  $B_1(r) = d_1 r + d_2$ ,  $\beta = \dot{R}_{11}/2R_{11}$ , and  $\lambda = \dot{R}_{22}/2R_{22}$ , it follows from eq.(12) that  $\lambda R_{22} = 0$ . Therefore, if  $\lambda \neq 0 \neq \beta$ ,  $R_{22} = 0$ , then we get

$$\begin{aligned}
\xi^1 &= -\beta \left( d_1 \frac{r^2}{2} + d_2 r \right) + d_1 \int \frac{\sqrt{|R_{44}|}}{R_{11}} + d_3, \\
\xi^2 &= -\lambda \theta (d_1 r + d_2), \\
\xi^4 &= \frac{d_1 r + d_2}{\sqrt{|R_{44}|}}.
\end{aligned}$$

If  $\lambda = 0$ ,  $\beta \neq 0 \neq R_{22}$ , then this case reduces to the subcase (viii.I.c.\*) and it follows  $R_{22} = \text{const.}$  If  $\lambda = 0 = \beta$ ,  $R_{22} \neq 0$ , then it reduces to the subcase (viii.I.c.†), and follows that  $R_{11}$  and  $R_{22}$  are constants.

**Case (xiv).**  $\xi = (0, \xi^2(x^a), \xi^3(x^a), \xi^4(x^a))$ .

In this case, after some algebra, it follows from RC equations that  $\xi^4$  vanishes, and  $R_{11}$  becomes constant. Therefore, this case is reduced to the case (vii).

## 4 Family of Contracted Ricci Collineations

For the Bianchi types I and III, and Kantowski-Sachs space-times, the family of CRC equation (2) takes the form

$$\frac{R_{11}}{A^2} \xi_{,1}^1 + \frac{R_{22}}{B^2} \left( \xi_{,2}^2 + \frac{f'}{f} \xi^2 + \xi_{,3}^3 \right) + \left( \frac{\dot{R}_{11}}{A^2} + 2 \frac{\dot{R}_{22}}{B^2} - \dot{R}_{44} \right) \xi^4 - R_{44} \xi_{,4}^4 = 0. \quad (39)$$

However, it is not possible to find a solution to (39) without imposing some additional restrictions either on the metric tensor, on the Ricci tensor, or on the vector fields  $\xi^a$ .



An example of proper the family of CRC is obtained by setting each of the components of  $\xi^a(x^b)$  to zero. Therefore, using the cases (i) and (iv) of the above section in eq. (39), we obtain the following equations, respectively,

$$\frac{R_{11}}{A^2}\xi_{,1}^1 = 0, \quad (40)$$

$$\frac{R_{22}}{B^2}\left(\xi_{,2}^2 + \frac{f'}{f}\xi^2\right) = 0, \quad (41)$$

$$\frac{R_{22}}{B^2}\xi_{,3}^3 = 0, \quad (42)$$

$$\left(\frac{\dot{R}_{11}}{A^2} + 2\frac{\dot{R}_{22}}{B^2} - \dot{R}_{44}\right)\xi^4 - R_{44}\xi_{,4}^4 = 0. \quad (43)$$

Now, for the case (i), using Eq. (40), we get the possibilities: (a.1) $R_{11} = 0$ , (b.1) $\xi_{,1}^1 = 0$ . In the first subcase,  $\xi^1$  is unconstrained, while in the second one,  $\xi^1 = \xi^1(\theta, \phi, t)$ . In case (ii), using Eq. (41), we find that (a.2) $R_{22} = 0$ , (b.2) $\xi_{,2}^2 + (f'/f)\xi^2 = 0$ . Therefore, in subcase (a.2),  $\xi^2$  is unconstrained, and in subcase (b.2), we obtain

$$\xi = F(r, \phi, t)f\partial/\partial\theta$$

where  $F(r, \phi, t)$  is an integration constant, and  $f$  take values  $\theta$ ,  $\sin\theta$ , and  $\sinh\theta$  for Bianchi I ( $\delta = 0$ ), Bianchi III ( $\delta = 1$ ), and Kantowski-Sachs ( $\delta = -1$ ), respectively. In case (iii), from Eq.(42), we have two different cases : (a.3) $R_{22} = 0$ , (b.3) $\xi_{,3}^3 = 0$ . Thus, in each subcases, we get that  $\xi^3$  is unconstrained, and  $\xi^3 = \xi^3(r, \theta, t)$ , respectively.

In case (iv), if we set  $R_{11}$  and  $R_{22}$  to constants, then, using Eq. (43), we get

$$\xi^4 = \frac{G(r, \theta, \phi)}{\sqrt{|R_{44}|}},$$

where  $G(r, \theta, \phi)$  is an integration constant. In the other cases, it is difficult to solve the equation (39).

## 5 Conclusion

In the present paper, the Bianchi types I and III, and Kantowski-Sachs space-times are classified according to their RCVs. In cases (i)-(iii), RCVs are

coincide with KVs, and then it follows that case (i) represents a translation along radial  $r$  direction, and the other cases (ii) and (iii) represent a rotation around  $\theta$  and  $\phi$  directions, respectively. The situation arising when any of  $R_{ii}$  for  $i = 1, 2, 4$  vanishes is identical to the fact that  $R_{ab}$  is degenerate, i.e.  $\det(R_{ab}) = 0$ , and the corresponding Lie algebra of RCVs is infinite dimensional. Further, in cases (ii) and (v), from (7), it is shown that  $\psi X$  is also a RC, because of  $R_{22}\xi^2 = 0$ . Also, in case (v), we have obtained that RCVs and KVs are identical if  $R_{ab}$  is nondegenerate, i.e.  $\det(R_{ab}) \neq 0$ , and RCVs are nontrivial (i.e., nonproper) if  $R_{ab}$  is degenerate. It should be noticed that (iv), (viii), (ix), (x), (xii), (xiii), and (xiv) types of symmetry vector  $\xi^a$  containing the component  $\xi^4$  would already lead to proper RCV, and the cases (vi) and (vii) which are not containing  $\xi^4$  are nonproper for this cosmological model. In case (xi), when  $\delta = 0$  (Bianchi type I), some of the RCVs are proper but when  $\delta = -1$  (Bianchi type III and Kantowski-Sachs), then RCVs are nonproper. As pointed out Hall *et al.* [13], in classification of the Bianchi types I and III, and Kantowski-Sachs space-times according to their RCVs, we obtain the following results from the discussion so far: (a) For this cosmological model the vector space of RCVs is infinite dimensional if  $R_{ab}$  is degenerate, and is finite dimensional if  $R_{ab}$  is nondegenerate. (b) It may properly contain the Killing or homothetic Lie algebra. For a proper Einstein space, the vector space of RCVs coincides with the Killing algebra. (c) Every KV is a RCV for the Bianchi types I and III, and Kantowski-Sachs space-times, but the converse is not true. Also, in case (v), some metrics are found under the time transformation, in which some of them (the cases with  $\delta = 0$ -Bianchi I) are known [3], and the others new. Further, we note that the cosmological metrics with two RCVs are important because of the cylindrical and plane metrics, and many of the Bianchi metrics, are special cases of  $G_2$  cosmologies [17]. Finally, in section 4, for the cases (i)-(iv), we find some proper family of CRC vector.

In this paper, the RC equations (8)-(17) have been obtained by assuming that  $R_{ii}$ , ( $i = 1, 2, 4$ ) do not vanish. However, one can provide a classification of the RCs according to the nature of the Ricci tensor. This case will be the subject of an another article.

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## References

- [1] G. H. Katzin, J. Levine, and W. R. Davis, *J. Math. Phys.* **10**, 617 (1969).
- [2] W. R. Davis, L. H. Green, and L. K. Norris, *Nuovo Cimento* **B34**, 256 (1976).
- [3] D. Kramer, H. Stephani, M. A. H. MacCallum, and E. Herlt, *Exact Solutions of Einstein Field Equations*, (VEB Deutscher Verlag der Wissenschaften, Berlin/Cambridge U.P., Cambridge, 1980).
- [4] L. H. Green, L. K. Norris, D. R. Jr. Oliver Jr., and W. R. Davis, *Gen.Rel.Grav.* **8**, 731 (1977).
- [5] L. Nunez, U. Percoco, and V. M. Villalba, *J. Math. Phys.* **31**, 137 (1990).
- [6] D. R. Oliver and W. R. Davis, *Gen. Rel. Grav.* **8**, 905 (1977).
- [7] S. Hojman, L. Nunez, A. Patino, and H. Rago, *J. Math. Phys.* **27**, 281 (1986).
- [8] G. H. Katzin and J. Levine, *Colloq. Math.(Poland)* **26**, 21 (1972).
- [9] G. H. Katzin and J. Levine, *J. Math. Phys.* **22**, 1878 (1981).
- [10] A. H. Bokhari and A. Qadir, *J. Math Phys.* **34**, 3541 (1993).
- [11] M. J. Amir, A. H. Bokhari, and A. Qadir, *J. Math. Phys.* **35**, 3005 (1994).
- [12] J. Carot, J. da Costa, and E. G. L. R. Vaz, *J. Math. Phys.* **35**, 4832 (1994).
- [13] G. S. Hall, I. Roy, and E. G. L. R. Vaz, *Gen. Rel. Grav.* **28**, 299 (1996).
- [14] İ. Yavuz and U. Camcı, *Gen. Rel. Grav.* **28**, 691 (1996).
- [15] M. A. H. MacCallum, *In General Relativity :An Einstein Centenary Survey*, S. W. Hawking and W. Israel (eds.), (Cambridge U.P., Cambridge, 1979), p.533.

- [16] H. Baofa, *Int. J. Theor. Phys.* **30**, 1121 (1991).
- [17] D. Lorenz, *J. Phys. A : Math. Gen.* **15**, 2809 (1982).
- [18] M. A. H. MacCallum, *The Origin of Structure in the Universe*, E. Gunzig and P. Nardone (eds.) (Kluwer Academic Press, Netherlands, 1991), p.131.